

Entropy of the Schwarzschild-de Sitter black hole due to arbitrary spin fields in different coordinates

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ABSTRACT: By using the Newman-Penrose formalism and the improved thin-layer “brick wall” approach, the statistical-mechanical entropies of the Schwarzschild-de Sitter black hole arising from quantum massless arbitrary spin fields are studied in the Painlevé and Lemaitre coordinates. Although the metrics in both the Painlevé and the Lemaitre coordinates do not obviously possess the singularities as that in the Schwarzschild-like coordinate, we find that, for arbitrary spin fields, the entropies in the Painlevé and Lemaitre coordinates are exactly equivalent to that in the Schwarzschild-like coordinate.

KEYWORDS: Black Holes, Spacetime Singularities.

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Contents

1. Introduction	1
2. Metrics of Schwarzschild-de Sitter spacetime in Painlevé and Lemaitre coordinates	3
2.1 Painlevé coordinate representation for Schwarzschild-de Sitter black hole	3
2.2 Lemaitre coordinate representation for Schwarzschild-de Sitter black hole	4
3. Entropy of Schwarzschild-de Sitter black hole due to arbitrary spin fields in different coordinates	4
3.1 Entropy of Schwarzschild-de Sitter black hole due to arbitrary spin fields in Painlevé coordinate	5
3.2 Entropy of Schwarzschild-de Sitter black hole due to arbitrary spin fields in Lemaitre coordinate	9
4. Summary	12

1. Introduction

Bekenstein and Hawking [1, 2] found that, by comparing black hole physics with thermodynamics and from the discovery of black hole evaporation, black hole entropy is proportional to the area of the event horizon. The discovery is one of the most profound in modern physics. However, the issue of the exact statistical origin of the black hole entropy has remained a challenging one. Recently, much effort has been concentrated on the problem [3]–[17]. The “brick wall” model (BWM) proposed by ’t Hooft [18] is an extensively used way to calculate the entropy in a variety of black holes, black branes, de Sitter spaces, and anti-de Sitter spaces [6]–[17]. In this model the Bekenstein-Hawking entropy of the black hole is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the event horizon. In refs. [6, 17], the authors found that although the original BWM has contributed a great deal to the understanding and calculation of the entropy of a black hole, there are some drawbacks in it such as the little mass approximation and taking the term including L^3 (L being the “infrared cutoff”) as a contribution of the vacuum surrounding the black hole, etc. The model is constructed on the basis of thermal equilibrium at a large scale, so it cannot be applied to cases out of thermal non-equilibrium problems, such as spacetime with two horizons, for example, a Schwarzschild-de Sitter black hole and Vaidya black hole [16, 17]. Therefore, in this paper we utilize the improved thin-layer BWM [6, 17] to resolve Schwarzschild-de Sitter spacetime.

In quantum field theory, we can use a timelike Killing vector to define particle states. Therefore, in static spacetimes we know that it is possible to define positive frequency modes by using the timelike Killing vector. However, in these spacetimes there could arise more than one timelike Killing vector which make the vacuum states inequivalent. This means that the concept of particles is not generally covariant and depends on the coordinate chosen to describe the particular spacetime. It is therefore interesting to study the following question: can we obtain the same statistical mechanical entropies of the black hole in the Painlevé and the Lemaitre coordinates due to the fact that in this question arises naturally after Shankaranarayanan *et al.* [19, 20] studied the Hawking temperature of the Schwarzschild black hole in the Painlevé and the Lemaitre coordinates by using the method of complex paths and they showed that the results are equal to those in the Schwarzschild-like coordinate. For the massless scalar field in the general static black hole, Jing [7] find that the entropies in the Painlevé and Lemaitre coordinates are exactly equivalent to that in the Schwarzschild-like coordinate. However, whether the entropies of the Schwarzschild-de Sitter black hole due to arbitrary spin fields are the same in the Painlevé, the Lemaitre and Schwarzschild-like coordinates is still an open question. In this paper we will study the question carefully.

In order to compare the results obtained in this article with the entropy of the Schwarzschild-de Sitter black hole in the Schwarzschild-like coordinate, we first introduce the entropy for the Schwarzschild-like coordinate. The Schwarzschild-de Sitter spacetime in the Schwarzschild-like coordinate is described by

$$ds^2 = f dt_s^2 - f^{-1} dr^2 - r^2 d\Omega^2, \tag{1.1}$$

with

$$f(r) = 1 - \frac{2m}{r} - \frac{\lambda}{3} r^2,$$

where m is the mass of the black hole and λ is cosmological constant which we will assume it is positive, and the geometric unit $G = c = \hbar = \kappa_B = 1$ is used. The Schwarzschild-de Sitter black hole have two horizons, i. e., the black hole event horizon r_H and the cosmological horizon r_C

$$r_C = \frac{2}{\sqrt{\lambda}} \cos \alpha, \quad r_H = \frac{2}{\sqrt{\lambda}} \cos \left(\alpha + \frac{\pi}{3} \right), \quad \text{with} \quad \alpha = \frac{1}{3} \arctan \sqrt{\frac{1}{9\lambda m^2} - 1}.$$

By using the improved thin-layer brick wall method, S. Q. Wu and M. L. Yan [6] found that the entropy of the Schwarzschild-de Sitter black hole due to arbitrary spin field in the Schwarzschild-like coordinate is

$$S/g_s = \frac{15 + (-1)^{2s}}{16} \left[\frac{A_h}{48\pi\epsilon^2} + \frac{1}{45} \left(1 - \frac{\lambda r_h^2}{2} \right) \ln \frac{\Lambda}{\epsilon} \right] - \frac{3 + (-1)^{2s}}{4} \frac{\lambda(1 + 2s^2)}{36\pi} A_h \ln \frac{\Lambda}{\epsilon}, \tag{1.2}$$

where $g_s = 1$ for scalar field ($s = 0$), $g_s = 2$ for Weyl neutrino ($s = 1/2$), Maxwell electromagnetic ($s = 1$), Rarita Schwinger gravitino ($s = 3/2$) and linearized Einstein gravitational ($s = 2$) fields, and $g_s = 4$ for massless Dirac field ($s = 1/2$), $A_h = 4\pi r_C^2$ or

$4\pi r_H^2$, respectively. From (1.2) we can know that the entropies depend not only on the spins of the particles but also on the cosmological constant except different spin fields obey different statistics.

The paper is organized as follows. In section 2 the metrics of the Schwarzschild-de Sitter black hole in the Painlevé and Lemaitre coordinates are introduced. In section 3 the entropies of the Schwarzschild-de Sitter black hole due to the arbitrary spin fields in the Painlevé and Lemaitre coordinates are investigated. Section 4 is devoted to a summary.

2. Metrics of Schwarzschild-de Sitter spacetime in Painlevé and Lemaitre coordinates

We now introduce the metrics of the Schwarzschild-de Sitter black hole in the Painlevé and Lemaitre coordinates.

2.1 Painlevé coordinate representation for Schwarzschild-de Sitter black hole

The time coordinate transformation from the Schwarzschild-like coordinate (1.1) to the Painlevé coordinate [7] is

$$t = t_s + \int \frac{\sqrt{1 - f(r)}}{f(r)} dr. \tag{2.1}$$

The radial and angular coordinates remain unchanged. With this transformation, the line element (1.1) becomes

$$ds^2 = g_{tt}dt^2 + 2g_{tr}dtdr + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2, \tag{2.2}$$

with

$$\begin{aligned} g_{tt} &= f(r), & g_{tr} &= g_{rt} = -\sqrt{1 - f(r)}, & g_{rr} &= -1, \\ g_{\theta\theta} &= -r^2, & g_{\varphi\varphi} &= -r^2 \sin^2 \theta. \end{aligned} \tag{2.3}$$

The inverse metric is

$$\begin{aligned} g^{tt} &= 1, & g^{tr} &= g^{rt} = -\sqrt{1 - f(r)}, \\ g^{rr} &= -f(r), & g^{\theta\theta} &= -\frac{1}{r^2}, & g^{\varphi\varphi} &= -\frac{1}{r^2 \sin^2 \theta}. \end{aligned} \tag{2.4}$$

The metric in the Painlevé coordinate has distinguishing features: (a) The spacetime is stationary but not static, so the time direction remains to be a Killing vector; (b) there is now no singularity at $f(r) = 0$, so the metric is regular at horizons of the black hole. That is to say, the coordinate complies with perspective of a free-falling observer, who is expected to experience nothing out of the ordinary upon passing through the event horizon. However, we will see next that the event and cosmological horizons manifests themselves as singularities in the expression for the semiclassical action.

2.2 Lemaitre coordinate representation for Schwarzschild-de Sitter black hole

The coordinates that transform from the Painlevé coordinate (2.2) to the Lemaitre coordinate (V, U, θ, φ) are given by

$$U = \tilde{r} - t, \quad V = \tilde{r} + t, \tag{2.5}$$

with

$$\tilde{r} = t + \int \frac{dr}{\sqrt{1 - f(r)}},$$

where t is the Painlevé time and V is the Lemaitre time. The angular coordinates θ and φ remain the same. The line element (2.2) in the new coordinate becomes

$$ds^2 = g_{VV}dV^2 + 2g_{VU}dVdU + g_{UU}dU^2 + \tilde{g}_{\theta\theta}d\theta^2 + \tilde{g}_{\varphi\varphi}d\varphi^2, \tag{2.6}$$

with

$$\begin{aligned} g_{VV} = g_{UU} &= \frac{1 - \tilde{f}}{4}, & g_{VU} = g_{UV} &= -\frac{\tilde{f} + 1}{4}, \\ \tilde{g}_{\theta\theta} &= -y, & \tilde{g}_{\varphi\varphi} &= -y \sin^2 \theta, \end{aligned} \tag{2.7}$$

where

$$\tilde{f}(U) = 1 - f(r), \quad y(U) = r^2.$$

The inverse metric is

$$\begin{aligned} g^{VV} = g^{UU} &= -\frac{1 - \tilde{f}}{\tilde{f}}, & g^{VU} = g^{UV} &= -\frac{\tilde{f} + 1}{\tilde{f}}, \\ \tilde{g}^{\theta\theta} &= -\frac{1}{y}, & \tilde{g}^{\varphi\varphi} &= -\frac{1}{y \sin^2 \theta}. \end{aligned} \tag{2.8}$$

We can see that the Lemaitre coordinate is time-dependant and the metric (2.6) has no coordinate singularity just as in the Painlevé coordinates. However, we will know that the event and cosmological horizons also manifests themselves as singularities in the expression for the semiclassical action.

3. Entropy of Schwarzschild-de Sitter black hole due to arbitrary spin fields in different coordinates

In this section we will study the entropy of the Schwarzschild-de Sitter black hole due to arbitrary spin fields in the Painlevé and Lemaitre coordinates by using the improved thin-layer brick wall method.

3.1 Entropy of Schwarzschild-de Sitter black hole due to arbitrary spin fields in Painlevé coordinate

Now in order to derive the master equation for arbitrary spin fields in Painlevé coordinate (2.2), we work it within the Newman-Penrose formalism [21, 22] by taking covariant components of the null tetrad vectors as

$$\begin{aligned} l_\mu &= \left(1, -\frac{1 + \sqrt{1 - f(r)}}{f(r)}, 0, 0\right), & n_\mu &= \frac{1}{2} \left(f(r), \frac{f(r)}{1 + \sqrt{1 - f(r)}}, 0, 0\right), \\ m_\mu &= -\frac{r}{\sqrt{2}}(0, 0, 1, i \sin \theta), & \bar{m}_\mu &= -\frac{r}{\sqrt{2}}(0, 0, 1, -i \sin \theta), \end{aligned} \quad (3.1)$$

The non-zero spin coefficients are

$$\begin{aligned} \rho &= -\frac{1}{r}, \\ \mu &= -\frac{1}{2r}f(r), \\ \gamma &= \frac{1}{4}f'(r), \\ \alpha = -\beta &= -\frac{1}{2\sqrt{2}r} \cot \theta, \end{aligned} \quad (3.2)$$

where a prime denotes the differential with respect to r , and only one non-zero Weyl tensor is

$$\Psi_2 = -\frac{m}{r^3} + \frac{\lambda}{3}. \quad (3.3)$$

Assuming that the azimuthal and time dependence of the perturbed fields will be of the form $e^{i(m\varphi - Et)}$, we find that the directional derivatives are

$$\begin{aligned} D &= l^\mu \partial_\mu = \mathcal{D}_0, & \Delta &= n^\mu \partial_\mu = -\frac{\Delta_r}{2r^2} \mathcal{D}_0^\dagger, \\ \delta &= m^\mu \partial_\mu = \frac{1}{\sqrt{2}r} \mathcal{L}_0^\dagger, & \bar{\delta} &= \bar{m}^\mu \partial_\mu = \frac{1}{\sqrt{2}r} \mathcal{L}_0, \end{aligned} \quad (3.4)$$

with

$$\begin{aligned} \mathcal{D}_n &= \frac{\partial}{\partial r} - \frac{iK_1}{\Delta_r} (1 + \sqrt{1 - f(r)}) + n \frac{\Delta'_r}{\Delta_r}, \\ \mathcal{L}_n &= \frac{\partial}{\partial \theta} - K_2 + n \cot \theta, \\ \mathcal{D}_n^\dagger &= \frac{\partial}{\partial r} + \frac{iK_1}{\Delta_r} (1 - \sqrt{1 - f(r)}) + n \frac{\Delta'_r}{\Delta_r}, \\ \mathcal{L}_n^\dagger &= \frac{\partial}{\partial \theta} + K_2 + n \cot \theta, \end{aligned} \quad (3.5)$$

where

$$\Delta_r = r^2 f(r), \quad K_1 = Er^2, \quad K_2 = -\frac{m}{\sin \theta}. \quad (3.6)$$

With the help of the Newman-Penrose formalism [21, 22], it can be shown that decoupled master equations controlling the perturbations of Schwarzschild-de Sitter black hole for massless arbitrary spin fields (i.e., scalar, Weyl neutrino, source-free Maxwell electromagnetic, Rarita-Schwinger gravitino, and the linearized Einstein gravitational fields) read [10, 23, 24]

$$\{[D - (2s - 1)\epsilon + \epsilon^* - 2s\rho - \rho^*](\Delta - 2s\gamma + \mu) - [\bar{\delta} - (2s - 1)\beta - \alpha^*](\bar{\delta} - 2s\alpha) - (s - 1)(2s - 1)\Psi_2\}\Phi_s = 0, \quad (3.7)$$

for $s = 1/2, 1, 3/2, 2$ and

$$\{[\Delta + (2s - 1)\gamma - \gamma^* + 2s\mu + \mu^*](D + 2s\epsilon - \rho) - [\bar{\delta} + (2s - 1)\alpha + \beta^*](\delta + 2s\beta) - (s - 1)(2s - 1)\Psi_2\}\Phi_{-s} = 0, \quad (3.8)$$

for $s = 0, -1/2, -1, -3/2, -2$.

Using eqs. (3.2), (3.3) and (3.4), eqs. (3.7) and (3.8) can be expressed as

$$\begin{aligned} & \frac{1}{r^2} \left\{ \Delta_r^{-s} \frac{\partial}{\partial r} \left(\Delta_r^{1+s} \frac{\partial}{\partial r} \right) - 2iK_1 \sqrt{1-f(r)} \frac{\partial}{\partial r} + \frac{f(r)K_1^2 - isK_1\Delta_r'(1+\sqrt{1-f(r)})}{\Delta_r} \right. \\ & \quad \left. + \frac{s}{2}\Delta_r'' + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - [K_2 - s \cot\theta]^2 + 4is\omega r - (4s^2 + 2) \frac{\lambda r^2}{3} \right\} \Phi_s = 0, \\ & \frac{1}{r^2} \left\{ \Delta_r^s \frac{\partial}{\partial r} \left(\Delta_r^{1-s} \frac{\partial}{\partial r} \right) - 2iK_1 \sqrt{1-f(r)} \frac{\partial}{\partial r} + \frac{f(r)K_1^2 + isK_1\Delta_r'(1+\sqrt{1-f(r)})}{\Delta_r} \right. \\ & \quad \left. - \frac{s}{2}\Delta_r'' + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - [K_2 + s \cot\theta]^2 - 4is\omega r - (4s^2 + 2) \frac{\lambda r^2}{3} \right\} (r^{2s}\Phi_{-s}) = 0. \end{aligned} \quad (3.9)$$

We can easily find that they are dual by interchanging $s = -s$. Thus one only needs to consider the case of positive spin state s , and obtain the results for the negative spin state $-s$ by substituting $s \rightarrow -s$. Two of equations (3.9) can be combined into the form of Teukolsky's master equation [23]

$$\begin{aligned} & \left\{ f(r) \frac{\partial^2}{\partial r^2} + \frac{(1+s)\Delta_r'}{r^2} \frac{\partial}{\partial r} - 2iE \sqrt{1-f(r)} \frac{\partial}{\partial r} \right. \\ & \quad \left. + \frac{1}{r^2} \frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2} \frac{\partial}{\partial\theta} + E^2 - \frac{m^2}{r^2 \sin^2\theta} - \frac{isE\Delta_r'}{\Delta_r} (1 + \sqrt{1-f(r)}) \right. \\ & \quad \left. + \frac{4isE}{r} - \frac{2sm \cot\theta}{r^2 \sin\theta} + \frac{s}{2r^2} \Delta_r'' - \frac{\lambda}{3} (4s^2 + 2) - \frac{s^2}{r^2} \cot^2\theta \right\} \tilde{\Phi}_s = 0. \end{aligned} \quad (3.10)$$

Now we can calculate the entropy due to arbitrary spin fields for the nonextreme Schwarzschild-de Sitter black hole in Painlevé coordinate by the thin-layer BWM. First we try to seek the total number of modes with energy less than E . In order to do this, we make use of the WKB approximation and substitute $\tilde{\Phi}_s \sim e^{iG(r, \theta)}$ into the above Teukolsky's master eq. (3.10), then we obtain

$$\begin{aligned} & f(r)k_r^2 - 2E \sqrt{1-f(r)}k_r + \frac{1}{r^2}k_\theta^2 - E^2 + \frac{m^2}{r^2 \sin^2\theta} + \frac{2sm}{r^2 \sin\theta} \cot\theta \\ & \quad + \left(\frac{s \cot\theta}{r} \right)^2 + \frac{\lambda}{3} (4s^2 + 2) - \frac{s}{2r^2} \Delta_r'' = 0, \end{aligned} \quad (3.11)$$

where $k_r = G_{,r}$ and $k_\theta = G_{,\theta}$ are the momentum of the particles moving in r and θ , respectively. In terms of the covariant metric components $g_{\mu\nu}$, eq. (3.11) can be rewritten as

$$f(r)k_r^2 - 2E\sqrt{1-f(r)}k_r - E^2 - \frac{k_\theta^2}{g_{\theta\theta}} - \frac{m^2}{g_{\varphi\varphi}} + H_s = 0, \quad (3.12)$$

where

$$H_s = \frac{2sm}{r^2 \sin \theta} \cot \theta + \frac{s^2}{r^2} \cot^2 \theta + \frac{\lambda}{3}(4s^2 + 2) - \frac{s}{2r^2} \Delta_r''. \quad (3.13)$$

The roots of the eq. (3.12) are

$$k_r^\pm = \frac{E\sqrt{1-f(r)} \pm \sqrt{E^2 - f(r) \left[-\frac{k_\theta^2}{g_{\theta\theta}} - \frac{m^2}{g_{\varphi\varphi}} + H_s \right]}}{f(r)}, \quad (3.14)$$

the sign ambiguity of the square root is related to the “out-going” (k_r^+) or “in-going” (k_r^-) particles, respectively. Here we utilize the average of the radial momentum (the minus before the k_r^- is caused by a different direction),

$$\tilde{k}_r = \frac{k_r^+ - k_r^-}{2} = \frac{1}{f(r)} \sqrt{E^2 - f(r) \left(-\frac{k_\theta^2}{g_{\theta\theta}} - \frac{m^2}{g_{\varphi\varphi}} + H_s \right)}. \quad (3.15)$$

So in this way, we take all kinds of particles into account. Eq. (3.15) can be rewritten as

$$\tilde{k}_r = \frac{1}{\sqrt{-g^{rr}}} \sqrt{\frac{E^2}{g_{tt}} - \left[-\frac{k_\theta^2}{g_{\theta\theta}} - \frac{(m+m_0)^2}{g_{\varphi\varphi}} + V_s \right]}, \quad (3.16)$$

with

$$m_0 = s \cos \theta, \quad V_s = \frac{\lambda}{3}(4s^2 + 2) - \frac{s}{2r^2} \Delta_r''. \quad (3.17)$$

The number of modes with E is equal to the number of states in this classical phase space [11]

$$\begin{aligned} n_h(E, s) &= \frac{1}{(2\pi)^3} \int dr d\theta d\varphi \int d\tilde{k}_r dk_\theta dm \\ &= \frac{1}{3\pi} \int d\theta \int_{r_h+\varepsilon}^{r_h+N\varepsilon} dr \frac{\sqrt{-g}}{(g_{tt})^2} [E^2 - g_{tt}V_s]^{3/2}, \end{aligned} \quad (3.18)$$

under the improved thin-layer BWM boundary conditions

$$\begin{aligned} \Phi(t, r, \theta, \varphi) &= 0 \quad \text{for } r < r_H + \varepsilon \quad \text{and } r > r_H + N\varepsilon, \\ \Phi(t, r, \theta, \varphi) &= 0 \quad \text{for } r < r_C - N\varepsilon \quad \text{and } r > r_C - \varepsilon, \end{aligned}$$

where $\varepsilon \ll r_H$ (or r_C), N is an arbitrary big integer which removes the infrared divergence. It is obviously that the location of the brick wall and the meaning of this wall in the Painlevé coordinate are the same as that in the Schwarzschild-like coordinate.

The integral is taken only over those values for which the square root in eq. (3.18) exists. Summing over the positive and negative spin states $\pm s$, we get the total states number

$$n_h(E) = \frac{g_s}{2}[n_h(E, s) + n_h(E, -s)] \approx \frac{g_s}{3\pi}[I_{1h}E^3 + 3I_{2h}E], \quad (3.19)$$

where I_{1h} (I_{2h}) represents I_{1H} (I_{2H}) for event horizon or I_{1C} (I_{2C}) for the cosmological horizon, and these quantities are given by

$$\begin{aligned} I_{1H} &= \int d\theta \int_{r_H+\varepsilon}^{r_H+N\varepsilon} dr \frac{\sqrt{-g}}{g_{tt}^2}, & I_{1C} &= \int d\theta \int_{r_C-N\varepsilon}^{r_C-\varepsilon} dr \frac{\sqrt{-g}}{g_{tt}^2}, \\ I_{2H} &= \int d\theta \int_{r_H+\varepsilon}^{r_H+N\varepsilon} dr \frac{\sqrt{-g}}{g_{tt}} \left[\frac{\lambda}{3}(2s^2 + 1) \right], & I_{2C} &= \int d\theta \int_{r_C-N\varepsilon}^{r_C-\varepsilon} dr \frac{\sqrt{-g}}{g_{tt}} \left[\frac{\lambda}{3}(2s^2 + 1) \right]. \end{aligned}$$

In the above, we have expanded eq. (3.19) in the high frequency approximation and introduced an appropriate degeneracy g_s for each species of particles. Accordingly, the free energy F at inverse Hawking temperature β can be expressed as

$$\begin{aligned} F_h &= - \int_0^\infty dE \frac{n_h(E)}{e^{\beta E} - (-1)^{2s}} \\ &= -g_s \left[2\zeta(4) \frac{15 + (-1)^{2s}}{16\pi\beta^4} I_{1h} + \zeta(2) \frac{3 + (-1)^{2s}}{4\pi\beta^2} I_{2h} \right], \end{aligned} \quad (3.20)$$

where $\zeta(n) = \sum_{k=1}^\infty 1/k^n$ is the Riemann zeta function, $\zeta(4) = \pi^4/90$, $\zeta(2) = \pi^2/6$, etc. We can now obtain the entropy of the Schwarzschild-de Sitter black hole due to arbitrary spin fields from the standard formula $S_h = \beta^2(\partial F_h/\partial\beta)$

$$S_h/g_s = \frac{15 + (-1)^{2s}}{16} \left[\frac{A_h}{48\pi\epsilon_h^2} + \frac{1}{45} \left(1 - \frac{\lambda r_h^2}{2} \right) \ln \frac{\Lambda_h}{\epsilon_h} \right] - \frac{3 + (-1)^{2s}}{4} \frac{\lambda(1 + 2s^2)}{36\pi} A_h \ln \frac{\Lambda_h}{\epsilon_h}, \quad (3.21)$$

where the ultraviolet cutoff ϵ_h and the infrared cutoff Λ_h have been set by $\eta_h^2 = 2\epsilon_h^2/15$ and $N = \Lambda_h^2/\epsilon_h^2$ [12, 13], the proper distance η_h from the event horizon to the inner brick wall is $\eta_H = \int_{r_H}^{r_H+\varepsilon} \sqrt{-g_{rr} + g_{tr}^2/g_{tt}} dr \approx 2r_H(\varepsilon/\Delta'_{r_H})^{1/2} = 2\sqrt{\varepsilon r_H/(1 - \lambda r_H^2)}$ and from the cosmological horizon to the brick wall is $\eta_C = \int_{r_C-\varepsilon}^{r_C} \sqrt{-g_{rr} + g_{tr}^2/g_{tt}} dr \approx 2\sqrt{\varepsilon r_C/(1 - \lambda r_C^2)}$, and $A_h = 4\pi r_H^2$ or $4\pi r_C^2$.

We find that eq. (3.21) is in agreement with Wu-Yan's result (1.2). That is to say, the entropy calculated in the Painlevé coordinate is exactly equal to that in the Schwarzschild-like coordinate.

By the equivalence principle and the standard quantum field theory in flat space, to construct a vacuum state for the massless scalar field in the Painlevé spacetime we should leave all the positive frequency modes empty. Kraus [25] pointed out that for the metric (2.2) it is convenient to work along a curve $dr + \sqrt{1 - g(r)}dt = 0$, then the condition is simply a positive frequency with respect to t near this curve. It is easy to prove that the modes used to calculate the entropy are essentially the same as that in the Schwarzschild-like coordinate.

3.2 Entropy of Schwarzschild-de Sitter black hole due to arbitrary spin fields in Lemaitre coordinate

Now we calculated the entropy of Schwarzschild-de Sitter black hole due to arbitrary spin field in Lemaitre coordinates.

For the metric (2.6), The null tetrad vectors can be expressed as

$$\begin{aligned}
 l_\mu &= \left(-\frac{1}{2}\sqrt{\tilde{f}}, \frac{\sqrt{\tilde{f}}(1-\sqrt{\tilde{f}})}{2(1+\sqrt{\tilde{f}})}, 0, 0 \right), & n_\mu &= \left(-\frac{(1-\tilde{f})}{4\sqrt{\tilde{f}}}, \frac{(1+\sqrt{\tilde{f}})^2}{4\sqrt{\tilde{f}}}, 0, 0 \right), \\
 m_\mu &= -\frac{\sqrt{y}}{\sqrt{2}}(0, 0, 1, i \sin \theta), & \bar{m}_\mu &= -\frac{\sqrt{y}}{\sqrt{2}}(0, 0, 1, -i \sin \theta).
 \end{aligned} \tag{3.22}$$

We find the non-zero spin coefficients

$$\begin{aligned}
 \rho &= -\frac{1}{\sqrt{y}}, \\
 \mu &= \frac{(1-\tilde{f})}{\sqrt{\tilde{f}}} \frac{2}{\sqrt{y}}, \\
 \gamma &= \frac{2}{\sqrt{y}} - \frac{2\lambda\sqrt{y}}{\tilde{f}} + 2\left(\frac{m}{y} - \frac{\lambda}{3}\sqrt{y}\right), \\
 \epsilon &= -\frac{2}{\sqrt{\tilde{f}}}\left(\frac{m}{y} - \frac{\lambda}{3}\sqrt{y}\right), \\
 \alpha &= -\beta = -\frac{1}{2\sqrt{2y}} \cot \theta,
 \end{aligned} \tag{3.23}$$

and only one non-zero Weyl tensor

$$\Psi_2 = -\frac{m}{y\sqrt{y}} + \frac{\lambda}{3}.$$

Assuming that the azimuthal and time dependence of the perturbed fields will be of the form $e^{i(m\varphi - EV)}$, we find that the directional derivatives are

$$\begin{aligned}
 D &= l^\mu \partial_\mu = \mathcal{D}_0, & \Delta &= n^\mu \partial_\mu = -\frac{\Delta_U}{2y} \mathcal{D}_0^\dagger, \\
 \delta &= m^\mu \partial_\mu = \frac{1}{\sqrt{2y}} \mathcal{L}_0^\dagger, & \bar{\delta} &= \bar{m}^\mu \partial_\mu = \frac{1}{\sqrt{2y}} \mathcal{L}_0,
 \end{aligned} \tag{3.24}$$

with

$$\begin{aligned}
 \mathcal{D}_n &= \frac{\partial}{\partial U} - \frac{iK_1}{\Delta_U} (1 - \sqrt{\tilde{f}})^2 + n \frac{\Delta'_U}{\Delta_U}, \\
 \mathcal{L}_n &= \frac{\partial}{\partial \theta} - K_2 + n \cot \theta, \\
 \mathcal{D}_n^\dagger &= \frac{1}{\tilde{f}} \frac{\partial}{\partial U} - \frac{iK_1}{\Delta_U} \frac{(1 + \sqrt{\tilde{f}})^2}{\tilde{f}} + n \frac{\Delta'_U}{\Delta_U}, \\
 \mathcal{L}_n^\dagger &= \frac{\partial}{\partial \theta} + K_2 + n \cot \theta,
 \end{aligned} \tag{3.25}$$

where

$$\Delta_U = y(1 - \tilde{f}), \quad K_1 = Ey, \quad K_2 = -\frac{m}{\sin \theta}. \quad (3.26)$$

Substituting (3.24) and (3.23) into (3.7) and (3.8), we can obtain the Teukolsky's master equation

$$\begin{aligned} & \left\{ (1 - \tilde{f}) \frac{1}{\tilde{f}} \frac{\partial^2}{\partial U^2} + \frac{(1+s)\Delta'_U}{y} \frac{\partial}{\partial U} - \frac{(1+\tilde{f})2iE}{\tilde{f}} \frac{\partial}{\partial U} \right. \\ & \quad + \frac{1}{y} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{y} \frac{\partial}{\partial \theta} - \frac{1-\tilde{f}}{\tilde{f}} E^2 - \frac{m^2}{y \sin^2 \theta} \\ & \quad - \frac{isE\Delta'_U}{\Delta_U} \left(1 - \sqrt{\tilde{f}}\right)^2 + \frac{4isE}{\sqrt{y}} - \frac{2sm \cot \theta}{y \sin \theta} + \frac{s}{2y} \\ & \quad \left. \Delta''_U - \frac{\lambda}{3}(4s^2 + 2) - \frac{s^2}{y} \cot^2 \theta \right\} \tilde{\Phi}_s = 0. \quad (3.27) \end{aligned}$$

Taking $\tilde{\Phi}_s \sim e^{iG(U, \theta)}$ into the above Teukolsky's master equation (3.27), we have

$$\frac{1-\tilde{f}}{\tilde{f}} k_U^2 - \frac{1+\tilde{f}}{\tilde{f}} 2Ek_U + \frac{1-\tilde{f}}{\tilde{f}} E^2 - \frac{k_\theta^2}{\tilde{g}_{\theta\theta}} - \frac{m^2}{\tilde{g}_{\varphi\varphi}} + H_s = 0, \quad (3.28)$$

with

$$H_s = \frac{2sm}{y \sin \theta} \cot \theta + \frac{s^2}{y} \cot^2 \theta + \frac{\lambda}{3}(4s^2 + 2) - \frac{s}{2y} \Delta''_U, \quad (3.29)$$

where $k_U = G_{,U}$ and $k_\theta = G_{,\theta}$ are the momentum of the particle moving in U and θ , respectively. We get the roots of k_U as

$$k_U^\pm = \frac{1+\tilde{f}}{(1-\tilde{f})} E \pm \sqrt{\frac{\tilde{f}}{1-\tilde{f}}} \sqrt{\frac{4E^2}{1-\tilde{f}} + \frac{k_\theta^2}{\tilde{g}_{\theta\theta}} + \frac{m^2}{\tilde{g}_{\varphi\varphi}} - H_s}, \quad (3.30)$$

the roots are related to the ‘‘out-going’’ (k_U^+) and ‘‘in-going’’ (k_U^-) particles, respectively.

Here, we make use of the average of the U -direction momentum (the minus before the k_U^- is caused by a different direction)

$$\tilde{k}_U = \frac{k_U^+ - k_U^-}{2} = \sqrt{\frac{\tilde{f}}{1-\tilde{f}}} \sqrt{\frac{4E^2}{1-\tilde{f}} + \frac{k_\theta^2}{\tilde{g}_{\theta\theta}} + \frac{m^2}{\tilde{g}_{\varphi\varphi}} - H_s}. \quad (3.31)$$

Eq. (3.31) can be rewritten as

$$\tilde{k}_U = \frac{1}{\sqrt{-g^{UU}}} \sqrt{\frac{E^2}{g_{VV}} + \frac{k_\theta^2}{\tilde{g}_{\theta\theta}} + \frac{(m+m_0)^2}{\tilde{g}_{\varphi\varphi}} - V_s}, \quad (3.32)$$

with

$$m_0 = s \cos \theta, \quad V_s = \frac{\lambda}{3}(4s^2 + 2) - \frac{s}{2y} \Delta''_U. \quad (3.33)$$

Summing over the positive and negative spin states $\pm s$, we get the total states number

$$n_h(E) = \frac{g_s}{2} [n_h(E, s) + n_h(E, -s)] \approx \frac{g_s}{3\pi} [I_{1h} E^3 + 3I_{2h} E], \quad (3.34)$$

with

$$\begin{aligned} I_{1H} &= \int d\theta \int_{U_H + \tilde{\varepsilon}}^{U_H + \tilde{N}\tilde{\varepsilon}} dU \frac{\sqrt{-\tilde{g}}}{(g_{VV})^2}, & I_{1C} &= \int d\theta \int_{U_C - \tilde{N}\tilde{\varepsilon}}^{U_C - \tilde{\varepsilon}} dU \frac{\sqrt{-\tilde{g}}}{(g_{VV})^2}, \\ I_{2H} &= \int d\theta \int_{U_H + \tilde{\varepsilon}}^{U_H + \tilde{N}\tilde{\varepsilon}} dU \frac{\sqrt{-\tilde{g}}}{g_{VV}} \left[\frac{\lambda}{3} (2s^2 + 1) \right], & I_{2C} &= \int d\theta \int_{U_C - \tilde{N}\tilde{\varepsilon}}^{U_C - \tilde{\varepsilon}} dU \frac{\sqrt{-\tilde{g}}}{g_{VV}} \left[\frac{\lambda}{3} (2s^2 + 1) \right], \end{aligned} \quad (3.35)$$

where

$$\begin{aligned} \sqrt{-\tilde{g}} &= \frac{1}{2} \sin \theta \sqrt{\frac{3m^2}{\lambda} \left[\frac{1 - 2e^{\sqrt{3}\lambda U} + e^{2\sqrt{3}\lambda U}}{e^{\sqrt{3}\lambda U}} \right] + \frac{\lambda}{3} \left[\frac{3m(1 - 2e^{\sqrt{3}\lambda U} + e^{2\sqrt{3}\lambda U})}{2\lambda e^{\sqrt{3}\lambda U}} \right]^2}, \\ g_{VV} &= \frac{1}{4} \left\{ 1 - 2m \left[\frac{3m(1 - 2e^{\sqrt{3}\lambda U} + e^{2\sqrt{3}\lambda U})}{2\lambda e^{\sqrt{3}\lambda U}} \right]^{-1/3} - \frac{\lambda}{3} \left[\frac{3m(1 - 2e^{\sqrt{3}\lambda U} + e^{2\sqrt{3}\lambda U})}{2\lambda e^{\sqrt{3}\lambda U}} \right]^{2/3} \right\}. \end{aligned}$$

In above calculation, we used the improved thin-layer BWM boundary conditions

$$\begin{aligned} \Phi(V, U, \theta, \varphi) &= 0 \quad \text{for } U < U_H + \tilde{\varepsilon} \quad \text{and } U > U_H + \tilde{N}\tilde{\varepsilon}, \\ \Phi(V, U, \theta, \varphi) &= 0 \quad \text{for } U < U_C - \tilde{N}\tilde{\varepsilon} \quad \text{and } U > U_C - \tilde{\varepsilon}, \end{aligned}$$

where $\tilde{\varepsilon} = 2/\sqrt{3\lambda} \ln [\sqrt{\lambda\varepsilon^3/6m} + \sqrt{\lambda\varepsilon^3/6m + 1}]$ which gives the relation between the location of the brick wall in the Lemaitre and Schwarzschild-like coordinates, \tilde{N} is an arbitrary big integer, and U_H and U_C are

$$\begin{aligned} U_H &= \frac{2}{\sqrt{3\lambda}} \ln \left[\sqrt{\frac{4 \cos^3(\alpha + \pi/3)}{\cos 3\alpha}} + \sqrt{\frac{4 \cos^3(\alpha + \pi/3)}{\cos 3\alpha} + 1} \right] \\ &= \frac{2}{\sqrt{3\lambda}} \ln \left[\sqrt{\frac{\lambda r_H^3}{6m}} + \sqrt{\frac{\lambda r_H^3}{6m} + 1} \right], \\ U_C &= \frac{2}{\sqrt{3\lambda}} \ln \left[\sqrt{\frac{4 \cos^3 \alpha}{\cos 3\alpha}} + \sqrt{\frac{4 \cos^3 \alpha}{\cos 3\alpha} + 1} \right] \\ &= \frac{2}{\sqrt{3\lambda}} \ln \left[\sqrt{\frac{\lambda r_C^3}{6m}} + \sqrt{\frac{\lambda r_C^3}{6m} + 1} \right], \end{aligned}$$

which correspond to the event and cosmological horizons of the Schwarzschild-de Sitter black hole.

Then, the free energy can be expressed as

$$\begin{aligned} F_h &= - \int_0^\infty dE \frac{n_h(E)}{e^{\beta E} - (-1)^{2s}} \\ &= -g_s \left[2\zeta(4) \frac{15 + (-1)^{2s}}{16\pi\beta^4} I_{1h} + \zeta(2) \frac{3 + (-1)^{2s}}{4\pi\beta^2} I_{2h} \right], \end{aligned} \quad (3.36)$$

We can now obtain the entropy of the Schwarzschild-de Sitter black hole due to arbitrary spin fields in Lemaitre coordinate as

$$S_h/g_s = \frac{15 + (-1)^{2s}}{16} \left[\frac{A_h}{48\pi\epsilon_h^2} + \frac{1}{45} \left(1 - \frac{\lambda r_h^2}{2} \right) \ln \frac{\Lambda_h}{\epsilon_h} \right] - \frac{3 + (-1)^{2s}}{4} \frac{\lambda(1 + 2s^2)}{36\pi} A_h \ln \frac{\Lambda_h}{\epsilon_h}, \quad (3.37)$$

where the ultraviolet cutoff ϵ_h and the infrared cutoff Λ_h have been set by $\eta_h^2 = 2\epsilon_h^2/15$ and $\tilde{N} = \Lambda_h^2/\epsilon_h^2$ [12, 13], the proper distance η_h from the event horizon to the inner brick wall is $\eta_H = \int_{U_H}^{U_H+\tilde{\epsilon}} \sqrt{-g_{UU} + g_{VU}^2/g_{VV}} dU \approx 2\sqrt{\epsilon r_H/(1 - \lambda r_H^2)}$ and from the cosmological horizon to the brick wall is $\eta_C \approx 2\sqrt{\epsilon r_C/(1 - \lambda r_C^2)}$, and $A_h = 4\pi r_H^2$ or $4\pi r_C^2$.

Comparing with eqs. (1.2) and (3.21), we find that it equals to the entropies calculated in the Painlevé and Schwarzschild-like coordinates.

From above discussions we find that although both the Painlevé and the Lemaitre spacetimes do not possess the singularity at the event and cosmological horizons, the entropies calculated in the Painlevé and the Lemaitre coordinates are equivalent to that calculated in the Schwarzschild-like coordinate. It is well known that the wave modes obtained by using semiclassical techniques, in general, are the exact modes of the quantum system in the asymptotic regions. Thus, if the asymptotic structure of the spacetime is the same for the two coordinates, then the semiclassical wave modes associated with these two coordinate systems will be the same. From eq. (2.5) we know that the differential relationship between the Lemaitre time V and the Painlevé time t can be expressed as $dV = dt + d\tilde{r} = 2dt + dr/\sqrt{1 - f(r)}$. Now let us also work along the curve $dr + \sqrt{1 - f(r)}dt = 0$, we obtain $dV = dt$. It is shown that the two definitions of positive frequency — with respect to V in the Lemaitre spacetime and with respect to t in the Painlevé spacetime — do coincide. Therefore, it should not be surprised at the entropies driven from the modes in the Lemaitre and Painlevé coordinates are the same.

4. Summary

We have studied the statistical-mechanical entropies arising from the quantum massless arbitrary spin fields in the Painlevé and Lemaitre coordinate representations of the Schwarzschild-de Sitter black hole using the improved thin-layer BWM. At first sight, we might have anticipated that the results are different from that of the Schwarzschild-like coordinate due to two reasons: (a) both the Painlevé and Lemaitre coordinate representations possess a distinguishing property — there are no singularities at $f(r) = 0$ so the metrics are regular at the event and cosmological horizons of black hole; (b) it is not obvious that the time V in the Lemaitre spacetime tends to the time t in the Painlevé spacetime. However, by comparing our results (3.21) and (3.37), which are worked out exactly, with the well-known result (1.2), we have found that in both these coordinate representations the entropies are the same as that in the standard Schwarzschild-like coordinate representation.

There are two reasons lead us to obtain the same results in the different coordinates. a) Although either the Painlevé or Lemaitre coordinate does not possess the singularity, the event and cosmological horizons manifests themselves as singularities in the action function

and then there could be particles production. Hence we can use the knowledge of the wave modes of the quantum field to calculate the statistical-mechanical entropies. b) When we construct a vacuum state for the massless arbitrary spin fields in the Painlevé spacetime we take the condition $dr + \sqrt{1-f(r)}dt = 0$, and then we find that the modes used to calculate the entropies in both the Painlevé and Lemaitre coordinates are exactly the same as that in the Schwarzschild-like coordinates since both V and t tend to the Schwarzschild time t_s as r goes to infinity under this condition. Therefore, it should not be a surprise that the entropies driven from the modes in the Lemaitre, Painlevé, and Schwarzschild-like coordinates are the same.

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